Graphs and Ribbon Graphs

Mary Griffiths and Stephen Huggett

Centre for Mathematical Sciences, Plymouth University

Motivation

We are trying to understand the behaviour of the Bollobás-Riordan polynomial invariant of ribbon graphs when we combine two ribbon graphs in certain simple ways, corresponding to the k-sums (denoted $\oplus_k$) and tensor products (denoted $\otimes$) of abstract graphs. This is inspired by the analogous work of Brylawski showing how the Tutte polynomial behaves under tensor products of abstract graphs.

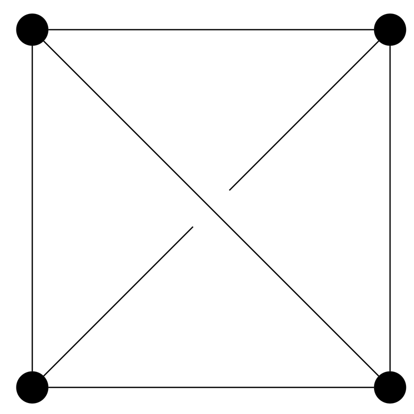
We are grateful to Robert Brignall for his help in producing this poster.

Introduction

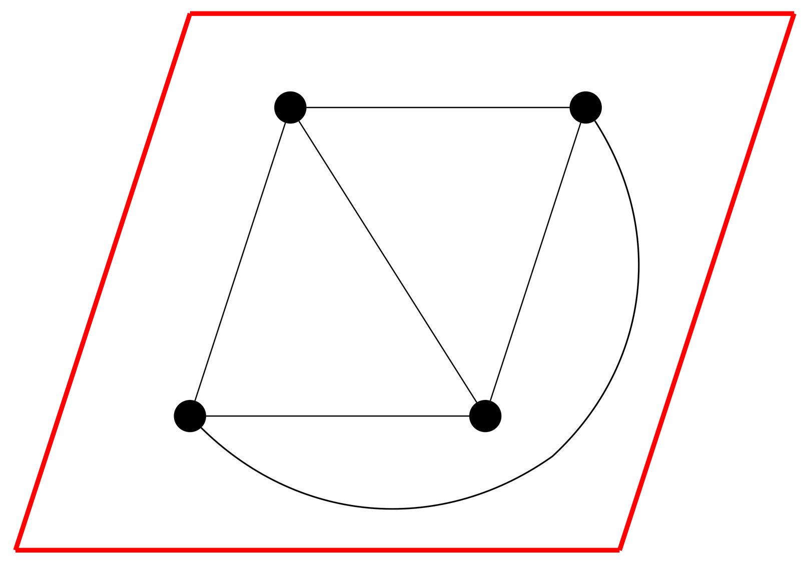
Abstract graphs are usually visualised as being drawn in space. But it is also of interest to embed them on a surface, such as a plane or a torus. Then, they are called ribbon graphs. The Tutte polynomial $T(G;x,y)$ is an extremely powerful invariant of abstract graphs G (although it is still far from being fully understood). For example, as a special case it includes the chromatic polynomial of G. The Tutte polynomial has been generalised to ribbon graphs, when it is called the Bollobás-Riordan polynomial.

These polynomials behave well under deletions or contractions of edges, and indeed they can be defined by this behaviour. Their behaviour under other operations on graphs is more subtle, though, and less well understood.

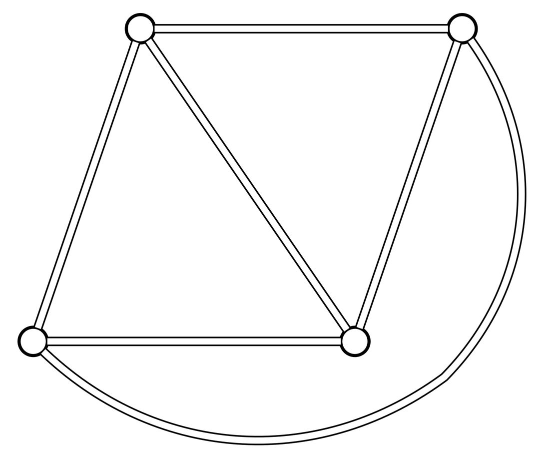
Graphs on surfaces



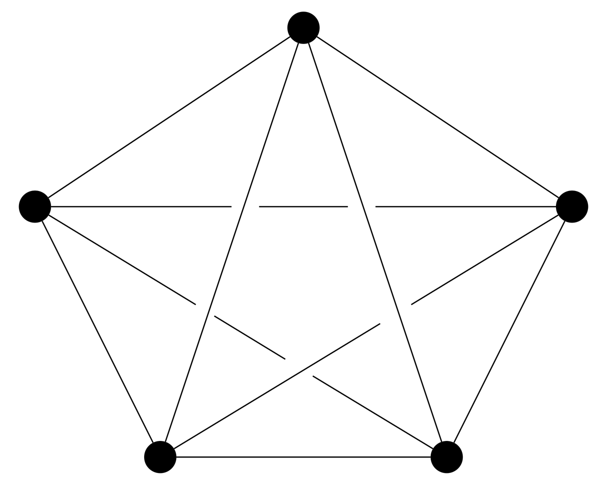
K_4 drawn in space



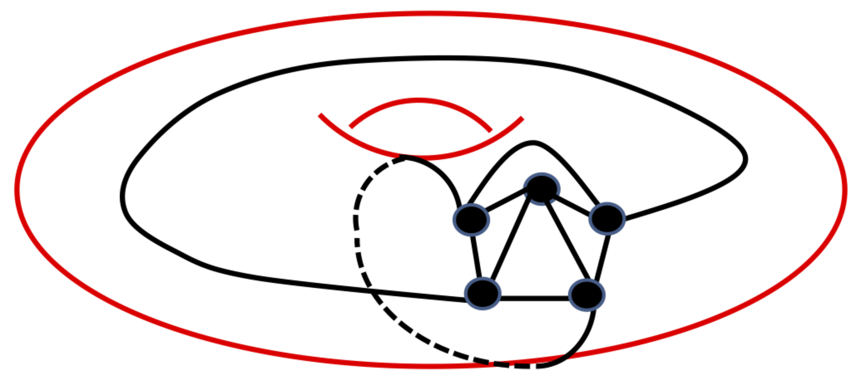
K_4 drawn on the plane



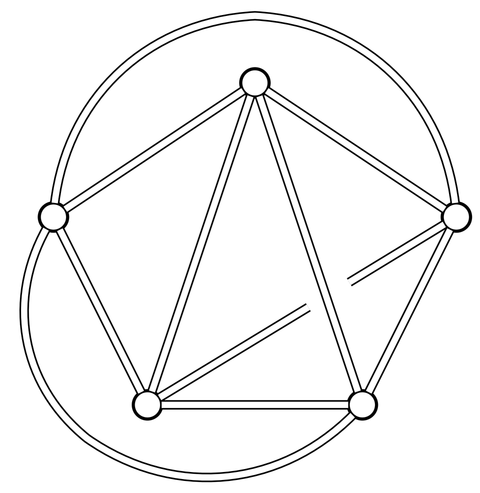
K_4 as a ribbon graph



K_5 drawn in space



K_5 drawn on a torus



K_5 as a ribbon graph

The Tutte polynomial

Let G be a multigraph, so it may have loops and multiple edges. Then the Tutte polynomial $T(G;x,y)$ can be defined recursively as follows. (To contract an edge, we simply delete it and then identify the two vertices at either end.)

For any edge e of G let G/e be the graph G with e contracted, and let $G\backslash e$ be the graph G with e deleted. Then

$$T(G;x,y) = \begin{cases} xT(G/e;x,y) & e \text{ a bridge} \\ yT(G\backslash e;x,y) & e \text{ a loop} \\ T(G\backslash e;x,y) + T(G/e;x,y) & \text{otherwise,} \end{cases}$$

while if G has no edges

$$T(G;x,y) = 1.$$

Now, given any multigraph G, we apply this recursion successively to its edges, in some order. It can be shown that the resulting polynomial in x and y—the Tutte polynomial of G—is independent of this order.

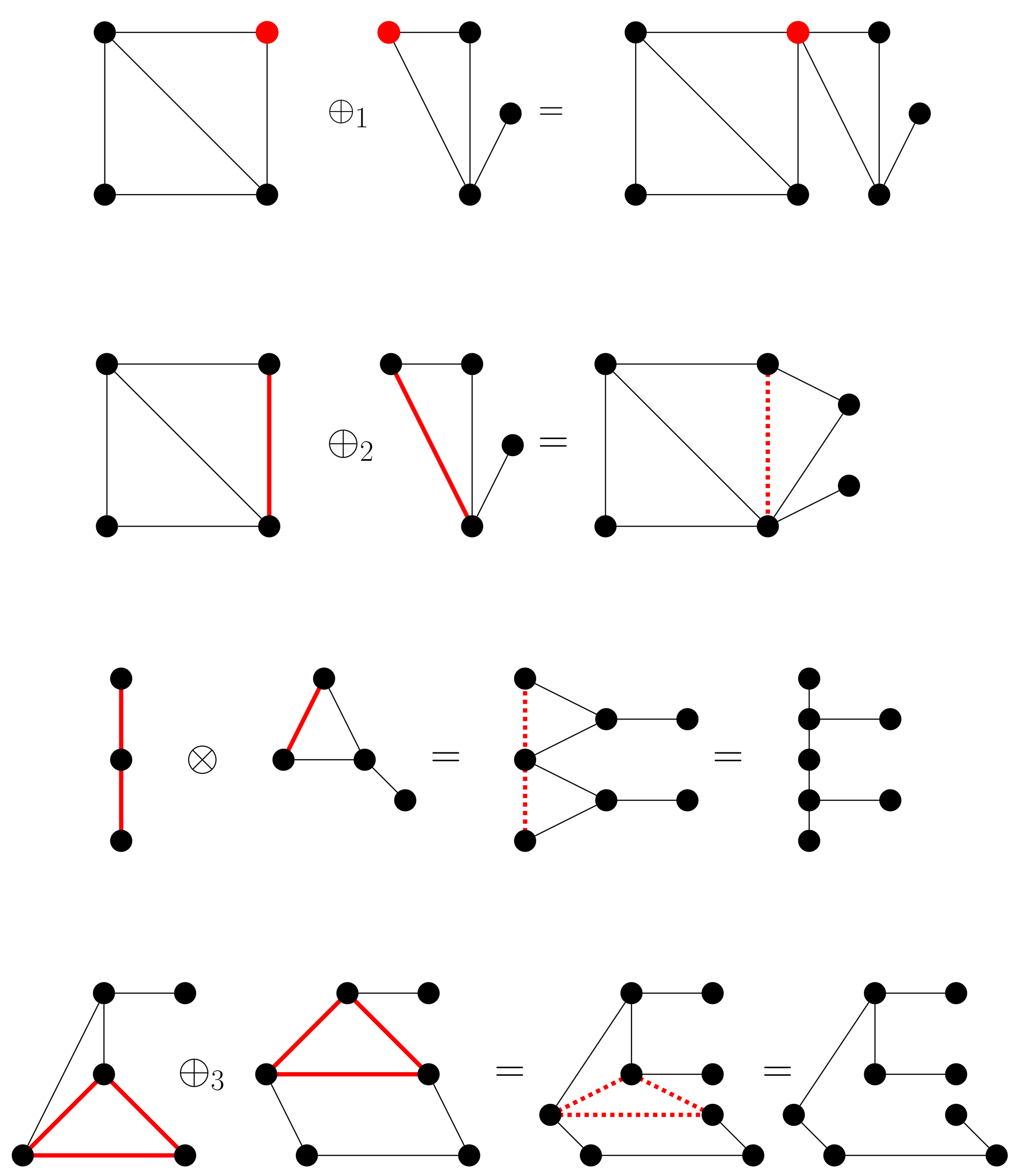
For example, let C be the graph consisting of a cycle of n vertices. Then

$$T(C;x,y) = y + x + x^2 + \cdots + x^{n-1}.$$

Now let C^* be the graph consisting of two vertices joined by n edges. It is the plane dual of C. Then

$$T(C^*;x,y) = x + y + y^2 + \cdots + y^{n-1}.$$

Sums and products of graphs



Brylawski's Theorem

Let $k(G)$ be the number of connected components of the graph G. Then its rank is

$$r(G) = |V(G)| - k(G)$$

and its nullity is

$$n(G) = |E(G)| - r(G).$$

Brylawski's Theorem states that the Tutte polynomial

$$T(G \otimes H;x,y)$$

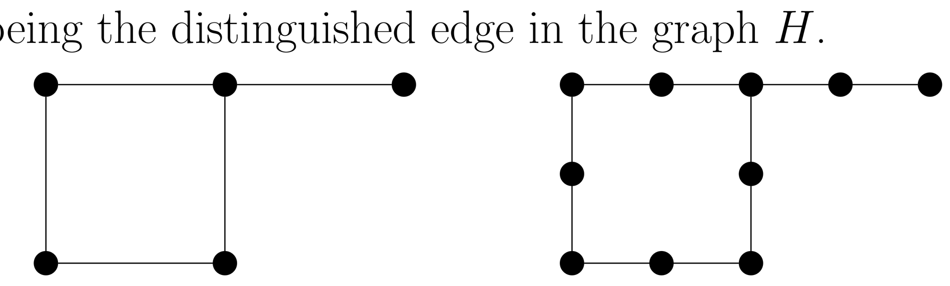
is equal to

$$\alpha^{n(G)}\beta^{r(G)}T\left(G;(x-1)\alpha/\beta+1, 1+(y-1)\beta/\alpha\right),$$

where α and β are the unique solutions to

$$(x-1)\alpha + \beta = T(H\backslash d;x,y)$$
$$\alpha + (y-1)\beta = T(H/d;x,y),$$

d being the distinguished edge in the graph H.



For example, if G is the graph on the left and $H = K_3$ then $G \otimes H$ is the graph on the right, and we can use Brylawski's result to obtain

$$T(G \otimes H, x, y) = (x+1)T(G, x^2, \frac{x+y}{x+1}).$$

This innocent looking result is of value in knot theory [1].

The Bollobás-Riordan polynomial

Let $f(G)$ be the number of boundary components of the ribbon graph G, and define $t(G)$ to be 0 if G is orientable and 1 otherwise. Then the Bollobás-Riordan polynomial is given by

$$R(G;x,y,z,w) = \sum_{A \subseteq E(G)} (x-1)^{r(G)-r(A)} y^{n(A)} z^{k(A)-f(A)+n(A)} w^{t(A)}.$$

This is a generalisation to ribbon graphs of the Tutte polynomial, because

$$R(G;x,y-1,1,1) = T(G;x,y),$$

and if G is embedded in the plane, then for all z and w

$$R(G;x,y-1,z,w) = T(G;x,y).$$

References

[1] Stephen Huggett *On tangles and matroids* Journal of Knot Theory and its Ramifications **14**, 7, 919–929, 2005.

[2] Stephen Huggett and Iain Moffatt *Expansions for the Bollobás-Riordan polynomial of separable ribbon graphs* Annals of Combinatorics **15**, 4, 675–706, 2011.