Entanglement and holography

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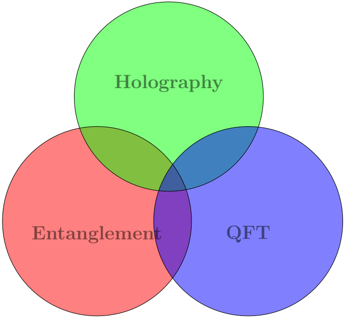
Introduction

New connections between entanglement (quantum information, condensed matter); quantum field theory (RG flow) and holography (black holes, quantum gravity).



Outline

- **Introduction to entanglement entropy**
- Holography and entanglement entropy
- Applications of holographic entanglement entropy

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Entanglement entropy

- Let ρ_A be the (reduced) density matrix of a quantum system.
- The entanglement entropy is the von Neumann entropy associated with ρ_A:

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A).$$

- Typically we will consider a bipartite partition of a Hilbert space, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and $\rho_A = \mathrm{Tr}_B(\rho)$.
- ρ is pure $\leftrightarrow S(\rho_A) = S(\rho_B)$.

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Example: 2 spin system

- Consider 2 spin system in pure state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

  where $|0\rangle$, $|1\rangle$ are up/down spin states.
- Then $\rho = |\psi\rangle\langle\psi|$ and tracing over second spin

$$\rho_A = \frac{1}{2}(|0\rangle_{AA}\langle 0| + |1\rangle_{AA}\langle 1|)$$

- Hence $S_A = S_B = \log 2$.

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Interpretation

Entanglement entropy counts the number of entangled bits between systems A and B.

Equivalently, e^{S_A} is the minimal number of auxiliary states needed to purify ρ_A.

Comments

- In quantum information, many other measures of entanglement are used e.g. measures of tripartite entanglement associated with partitioning a system into three.
- Entanglement entropy is however the most robust measure which is additive:

$$S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$$

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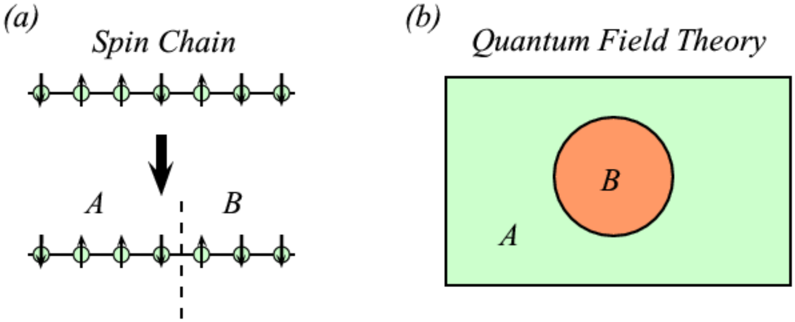
Geometric entanglement entropy

Typically, systems A and B correspond to subregions of space:



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Discrete systems

- Entanglement entropy can often be computed for spin chains using integrability.
- For ground states in gapped systems:

$$S_A \sim \mathrm{Area}(\partial A)$$

  i.e. scales with area of boundary between A and B.
- Intuitively: entanglement between spin pairs at boundary dominates.

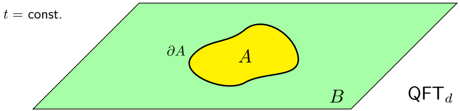
Discrete systems: quantum critical/CFT

- For a $(1+1)$-dimensional theory quantum critical theory in its ground state (Holzhey, Larsen, Wilczek):

$$S_A \sim c \log(L/\epsilon)$$

  where L is the width of the entangling region and c is the central charge of the 2d CFT.
- Here ϵ is the lattice spacing in the underlying discrete system.

Quantum critical in general dimensions



- For quantum critical theories in $(D+1)$ dimensions:

$$S_A \sim \frac{\text{Area}(\partial A)}{\epsilon^{D-1}} + \cdots + a \log(L/\epsilon)$$

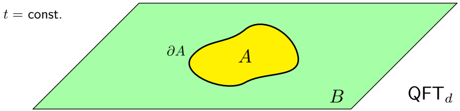
  where L is the scale of region A.

- Here a is the coefficient of the a anomaly in $\langle T^i_i \rangle$ for D odd.

Entanglement entropy in CMT/quantum information

1. Related to efficient computation of ground state wavefunctions (matrix product states etc).
2. Characterization of different phases (gapped, quantum critical etc).
3. Measurable in small cold atom systems?

Quantum field theory



- In $(D+1)$ dimensions:

$$S_A \sim c_{D-1}\frac{L^{D-1}}{\epsilon^{D-1}} + c_{D-2}\frac{L^{D-2}}{\epsilon^{D-2}} + \cdots + c_0 \log(L/\epsilon) + c_1$$

where L is the scale of region A and $\epsilon \sim 1/\Lambda$ is the UV cutoff.

Conceptual issues

- Entangement entropy is hard to compute outside free (integrable) theories.
- In practice it is usually computed as

$$S_A = \mathcal{L}_{n \to 1} \left(\frac{1}{1-n} \log(\mathrm{Tr}(\rho^n)) \right) = \mathcal{L}_{n \to 1} \left(S_A(n) \right)$$

  Note that the Rényi entropies $S_A(n)$ are not observables.
- Computation in gauge theories is problematic: Hilbert space does not factorise.

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Conceptual issues

S_A is UV divergent.

- Power law divergences are not present in zeta function or dimensional regularisation; comparison with discrete systems in CMT requires use of direct cutoff.
- Coefficients of logarithmic terms are sometimes called "universal".

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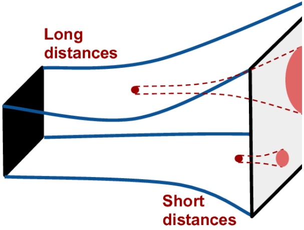
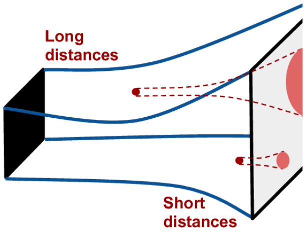
Holography



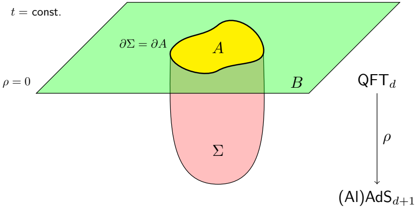
- QFT in d dimensions captured by gravity in $(d+1)$ dimensions.
- E.g. CFT_d captured by AdS_{d+1}.
- Holographic realizations of explicitly and spontaneously broken symmetry, confinement etc.

Holography



- UV physics near boundary, IR physics in deep interior.
- Holographic computables: partition function, correlation functions of gauge invariant operators etc

Holographic entanglement entropy



- Holographic Ryu-Takayanagi (RT) prescription: area of co-dimension two minimal surface Σ homologous to A

$$S_A = \frac{\mathcal{A}}{4G}$$

- QFT_3 - 2d minimal surface in 4d bulk.

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Holographic entanglement entropy: AdS

- Write the AdS metric as

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}\left(-dt^2 + dx \cdot dx\right)$$

and cut off the boundary at $\rho = \epsilon \ll 1$.

- For a surface of constant t, homologous to region A:

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{\epsilon^{D-1}} + \cdots + a \log(L/\epsilon)$$

as expected in a CFT.

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Geometric computation of entanglement entropy

Why is entanglement entropy computed by minimal surfaces?

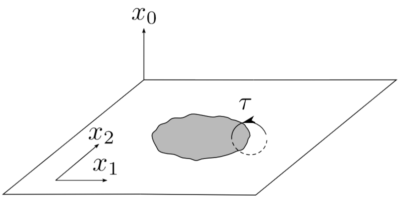
Derivation using replica trick

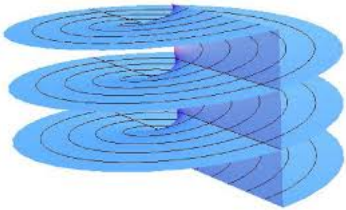
- We can rewrite the $n = 1$ Rényi entropy as:

$$S_A = -n\partial_n \left[\log Z(n) - n \log Z(1)\right]_{n=1}$$

- Here $Z(1)$ is the usual partition function.
- $Z(n)$: partition function on a replica space in which the circle around ∂A has periodicity $2\pi n$.

Replica trick





3d field theory: on replica space τ has periodicity $2\pi n$.

Visualisation of (x_0, x_1) slice of $n = 3$ replica space.

Derivation of Ryu-Takayanagi

- The defining holographic relation is

$$I_E = -\log Z$$

where I_E is the onshell gravity action

$$I_E = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{g}\,(R + 2\Lambda + \cdots)\,.$$

- Replica has conical singularity so

$$R(n) = R + 4\pi\delta_{\Sigma}$$

i.e. contribution localised on $(d-1)$-dim surface Σ.
(Solodukhin)

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Lewkowycz-Maldacena derivation of Ryu-Takayanagi

- Then all terms cancel in

$$S_A = n\partial_n \left[I_E(n) - n I_E(1) \right]_{n=1}$$

apart from the contribution from the surface Σ, giving

$$S_A = \frac{\mathcal{A}}{4G}$$

- Similar derivations for higher derivative gravity theories, matter coupled to curvature etc.

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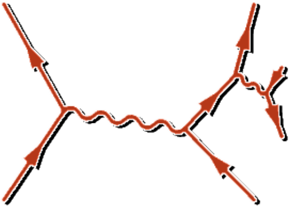
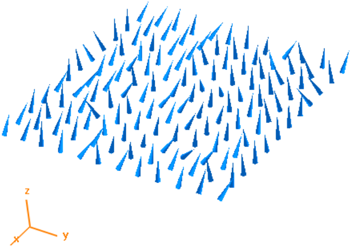
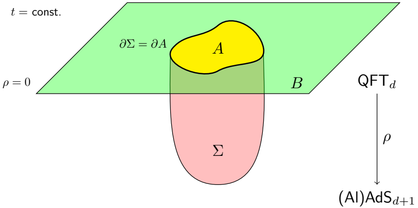
Significance of entanglement entropy





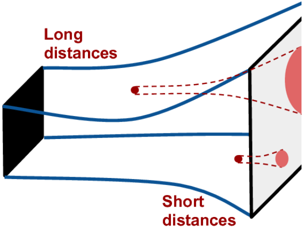
Why is holographic entanglement entropy useful for QFT/CMT?

Significance of holographic EE



- Simple prescription for computing EE in (large N, strongly coupled) gauge theories, strongly coupled superfluids etc.

Significance of entanglement entropy



- Small entangling regions probe UV physics; large regions probe IR physics.
- EE can probe IR behaviour of the theory (confinement etc). (Huijse et al; Klebanov et al; $\cdots$)
- Similar to Wilson loops, computed holographically via 2d minimal surfaces.

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Entanglement entropy in CFTs

- Recall universal terms for ground state of CFT in even dimensions

$$S_A \sim a \log\left(\frac{L}{\epsilon}\right)$$

- In $d = 2$, the trace anomaly is

$$\langle T^i_i \rangle = \frac{c}{12}\,\mathcal{R}$$

where $\mathcal{R}$ is the Ricci scalar of the background.

- Zamoldchikov c-theorem: c function monotonically decreases along RG flow, stationary at fixed points.

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Odd-dimensional CFTs

- In odd dimensions, there is no trace anomaly: $\langle T^i_i \rangle = 0$.
- Long-standing puzzle: what quantity measures the decrease in DoF along RG flow?

F quantity in 3d CFTs

- In a 3d CFT we define the F quantity as

$$F = -\ln Z_{S^3}$$

  and the F theorem states that $F_{UV} \geq F_{IR}$.

More precisely:

- F is conjectured to be stationary at the fixed point, positive in a unitary CFT and to decrease along any RG flow.

(Jafferis, Klebanov, Pufu, Safdi, ...)

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Conformal maps

The sphere partition function and the entanglement entropy for a disk entangling region in flat space are related by Casini-Huerta-Myers conformal map.

CHM map

- Starting from

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2$$

let

$$t = R \frac{\cos\theta \sinh \tau/R}{(1 + \cos\theta \cosh \tau/R)}$$

$$r = R \frac{\sin\theta}{(1 + \cos\theta \cosh \tau/R)}$$

so maps to de Sitter:

$$ds^2 = \Omega^2 (-\cos^2\theta d\tau^2 + R^2 (d\theta^2 + \sin^2\theta d\phi^2))$$

- Covers $0 \leq r < R$ in original coordinates, i.e. disk.

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Partition function and EE

- State in de Sitter is thermal with $\beta = 2\pi R$.
- EE for disk mapped to thermodynamic entropy, which in turn is related to the partition function

$$S_{\rm deSitter} = -W = \ln Z$$

- Disk entanglement entropy is thus proportional to the partition function on S^3.

Puzzle

Unlike the *a* anomaly coefficients, both F (S^3 partition function) and EE are UV divergent.

- Implicitly, F refers to the renormalized partition function, but what about scheme dependence?

Scheme independence of renormalized F and EE

1. The finite part of F is renormalization scheme independent (i.e. no finite counterterms exist).
2. One can define a renormalized entanglement entropy.

(M.T. and Woodhead)

Holographic computation of F

- The renormalized bulk action for asymptotically AdS_4 manifolds is (de Haro et al)

$$I = -\frac{1}{16\pi G}\int d^4x\sqrt{g}(R+6) + \frac{1}{8\pi G}\int d^3x\sqrt{h}(1-\frac{\mathcal{R}}{4})$$

  There are no possible finite counterterms.
- The renormalized onshell action for AdS_4 with S^3 slicing is

$$I = -\frac{\pi}{2G} \qquad \rightarrow \qquad F = \frac{\pi}{2G}.$$

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Renormalized entanglement entropy

- Recall the EE is:

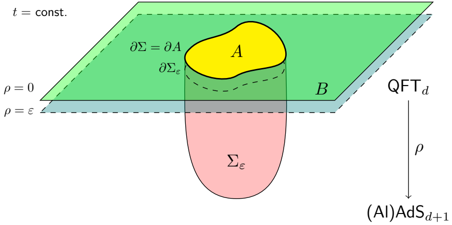
$$S_A = -n\partial_n \left[\log Z(n) - n \log Z(1)\right]_{n=1}$$

- The renormalized EE is computed via renormalized partition functions:

$$S_{\rm ren} = -n\partial_n \left[\log Z_{\rm ren}(n) - n \log Z_{\rm ren}(1)\right]_{n=1}$$

using any renormalization scheme.

Area renormalization



- Cutoff surface at $\rho = \epsilon$.
- Renormalize area using appropriately covariant counterterms.

Earlier work on renormalized minimal surfaces:
(Henningson/Skenderis; Graham/Witten; Gross et al)

Results for 3d CFT

- The renormalized EE in AdS_4 is

$$S_{\mathrm{ren}} = \frac{1}{4G} \int_{\Sigma} d^2 x \sqrt{\gamma} - \frac{1}{4G} \int_{\partial\Sigma} dx \sqrt{h}$$

with $\partial\Sigma$ the boundary of the minimal surface.

- No possible finite counterterms.

Disk entangling region

- Consider an entangling region which is a disk of radius R:

$$S_{\rm ren} = -\frac{\pi}{2G}$$

- Satisfies CHM relation

$$F = -S_{\rm ren}$$

NB Positivity of F requires negativity of $S_{\rm ren}$ for disk region.

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Behaviour of F under RG flows

- Under a relevant deformation of the 3d CFT

$$I \rightarrow I_{\text{CFT}} + \int d^3x \; \lambda \mathcal{O}_\Delta$$

the F quantity should decrease:

$$F(\lambda) = F_{UV} - \lambda^2 F_{(2)} + \mathcal{O}(\lambda^3)$$

with $F_{(2)} > 0$.

- And according to the CHM map:

$$\delta S_{\text{ren}} = \lambda^2 F_{(2)} + \mathcal{O}(\lambda^3)$$

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Evidence for F theorem

- Considerable evidence for $F_{UV} \geq F_{IR}$, in perturbative and holographic examples.
- Less evidence for monotonic decrease of F.

Change of F quantity under holographic RG flows

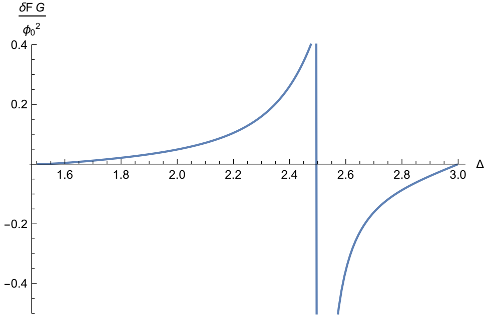
- Holographically a curved domain wall solution

$$ds^2 = dw^2 + e^{2A(w)} ds^2_{S^3} \qquad \phi(w)$$

corresponds to RG flow on S^3.

- Here the mass of the scalar ϕ corresponds to the dimension Δ of deforming operator $\mathcal{O}$.

Change of F quantity under holographic RG flows



- δF is positive for operator dimensions $\frac{3}{2} \leq \Delta < \frac{5}{2}$!
- $\Delta = 5/2$ answer is scheme dependent.

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Violation of strong F theorem

- Flows end at IR fixed point, for which $F_{IR} < F_{UV}$.
- Analytic argument why F increases, for range of bulk scalar masses.

Either these holographic flows are unphysical, or F does not decrease monotonically.

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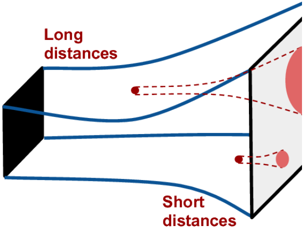
Entanglement entropy: alternatives to F theorem?



- $S_{\rm ren}$ also does not change monotonically along flow.
- But for disk regions of radius R we can show that

$$\frac{\partial^2 S_{\rm ren}}{\partial R^2} \leq 0$$

(Casini, Huerta, Myers) follows from subadditivity.

Summary

1. Entanglement entropy can be computed holographically in strongly coupled gauge theories.
2. In QFT entanglement entropy can be renormalized.
3. Gradients of the renormalized EE capture reduction in DoF along RG flow.

Conclusions and outlook

Enormous progress on understanding EE in QFT. Defining entanglement entropy in quantum gravity still very hard!

- Is (renormalized) EE in QFT a useful computable for high energy physicists? Computable on the lattice?
- Can we find a workable definition of EE in quantum gravity?

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