Relativistic Charge Dynamics in Electromagnetic Fields

Lauren Adams, Rajan Kandasamy and Tom Heinzl

Centre for Mathematical Sciences, Plymouth University

Motivation

A convenient starting point for the analysis of laser matter interactions is to determine the classical motion of a single charge (such as an electron) in external electromagnetic fields. This minimal physical system can then be extended in various ways by including: (i) radiation (reaction), (ii) many-particle interactions, and (iii) quantum effects, both for charges and fields.

It turns out, though, that the classical particle dynamics is a good starting point. For many systems of interest, classical integrability (if present) extends to the quantum situation (in a WKB type sense). This then allows for a non-perturbative, all-order inclusion of the external field into modified 'free' dynamics (adopting the Furry picture). With this in mind, we will exclusively focus here on classical motion and its integrability features.

Introduction

We consider a relativistic particle (charge e, mass m) coupled to an electromagnetic field encoded in the vector potential A_μ, then interaction described by the action

$$S = \int d\tau \left(-mc^2 - \frac{e}{c} A \cdot \dot{x} \right) \equiv \int d\tau L \; , \qquad (1)$$

a proper time integral of the Lagrangian L. Note that the field A does not have dynamics of its own, hence it is *external*. It is this field that will be used to model an intense laser beam of dimensionless strength $a_0 = e\langle -A^2 \rangle^{1/2}/mc^2$. (Henceforth: $c = 1$.)

Variation of the action with respect to x^μ yields the **Lorentz equation** of motion in covariant form,

$$m\ddot{x}^\mu = eF^{\mu\nu}(x)\dot{x}_\nu \equiv F^\mu \; , \qquad (2)$$

hence valid in all Lorentz frames, with the Einstein-Lorentz force F^μ given in terms of the field strength tensor, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. An invariant characterisation of the field is provided by the two basic invariants

$$\mathcal{S} := -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = \frac{E^2 - B^2}{2} \; , \; \mathcal{P} := -\frac{1}{4} F_{\mu\nu}\tilde{F}^{\mu\nu} = \mathbf{E}\cdot\mathbf{B} \; , \qquad (3)$$

where the tilde denotes the dual field strength. We will see in a moment that these invariants also help to characterise some charge trajectories.

Our immediate task is to solve the Lorentz equation (2) given A^μ, hence $F^{\mu\nu}$. All difficulty resides in the dependence of the field strength $F^{\mu\nu}$ on the trajectory, $x = x(\tau)$, which in general will render the equation nonlinear. To make analytical progress, we need to make simplifications, the most drastic one being to assume fields that are *constant* throughout space-time (or very nearly so).

Constant fields

When $F^{\mu\nu}$ is constant, we rewrite the Lorentz equation of motion (2) for momentum $p = m\dot{x}$ in matrix form as $\dot{p} = \mathbb{F}p$ (absorbing constants into $\mathbb{F}$). This is linear with the obvious integral

$$p(\tau) = \exp(\mathbb{F}\tau)\, p_0 \equiv \Lambda\, p_0 \; . \qquad (4)$$

It is crucial to note that $\mathbb{F}$ is antisymmetric, $\mathbb{F}^T = -\mathbb{F}$, whence the $\Lambda(\tau)$ can be interpreted as a Lorentz transformation. As a result, we find that:

Result 1: *Charge orbits in constant electromagnetic fields are in one-to-one correspondence with symmetry orbits ('conjugacy classes') of the Lorentz group.*

This is tantamount to cataloguing the eigenvalues of the field strength $\mathbb{F}$, which results in Taub's orbit classification [1] in terms of the invariants (3). The hodographs $p^\mu(\tau)$ yield conic sections (or a combination thereof), which easily lend themselves to a second integration. The resulting orbits are summarised in Fig. 1.

Constant fields: Orbits

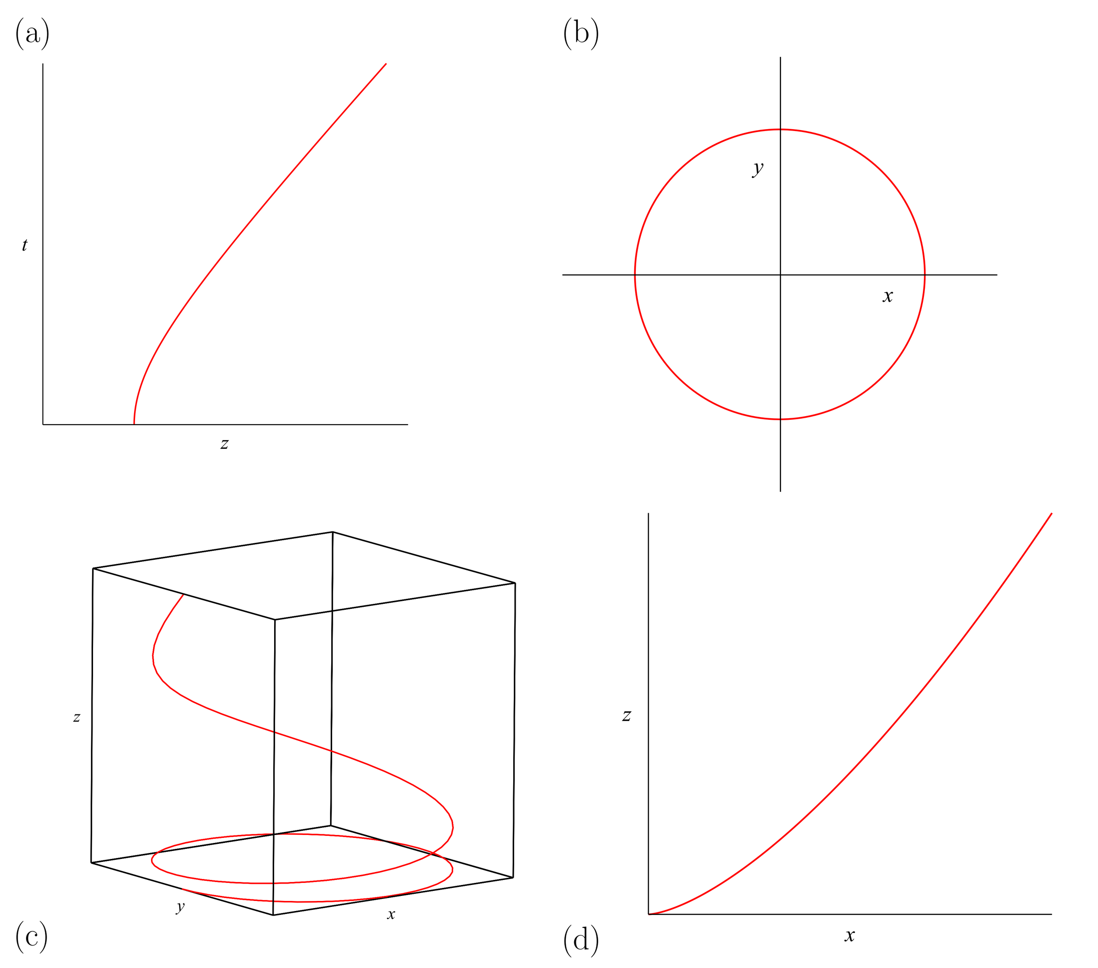

Fig. 1: Taub's classification of charge orbits
(a) Electric field: Hyperbolic motion, invariants $\mathcal{S} > 0$, $\mathcal{P} = 0$; (b) Magnetic field: Elliptic motion, invariants $\mathcal{S} < 0$, $\mathcal{P} = 0$; (c) Combined electric and magnetic fields: Loxodromic motion, invariants $\mathcal{S} \neq 0$, $\mathcal{P} \neq 0$; (d) Null (crossed) field: Parabolic motion, invariants $\mathcal{S} = 0$, $\mathcal{P} = 0$.

Univariate fields

The next simplest case is provided by fields that depend on a *single* space-time coordinate only. Such fields, which we call 'univariate', have the general form $F^{\mu\nu} = F^{\mu\nu}(l \cdot x)$, with l a constant four-vector. Hence, $l \cdot x$ may represent any of the four Minkowski coordinates or a linear combination thereof. The sign of l^2 then yields a natural classification in terms of the invariant Minkowski sub-spaces: time-like ($l^2 > 0$), space-like ($l^2 < 0$) or light-like ($l^2 = 0$).

In any case, it is obvious that (in 3+1 dimensions), a univariate field is invariant under three translations, namely in the three directions orthogonal to $l \cdot x$. Accordingly, out of the four canonical momenta,

$$\pi^\mu = \frac{\partial L}{\partial \dot{x}_\mu} = p^\mu + eA^\mu \; , \qquad (5)$$

three (linear combinations) must be conserved. Together with the mass shell constraint, $p^2 = m^2$, there are thus four conservation laws, whereupon we conclude:

Result 2: *For univariate fields there are four momentum conservation laws. Hence, there are four first integrals, and the momentum dynamics is integrable.*

Example: Plane wave

A laser beam is often modelled by a (possibly pulsed) plane wave (PW) field, which corresponds to the choice $l \equiv k$, a light-like wave vector defining the wave phase $\phi = k \cdot x$. For propagation along z, the vector potential is $A_\mu = A_\mu(x^-)$, a function of the single light-front coordinate $x^- = \phi/\omega = t - z$ with $\omega = |\mathbf{k}|$ denoting the wave frequency. This implies the three momentum conservation laws

$$\pi^- = p_0^- \; , \quad \pi^i = p_0^i + eA_0^i \; , \quad i = 1, 2 \; , \qquad (6)$$

where we have chosen light-front gauge, $A^- = 0$, for simplicity. Integrability follows immediately with the result

$$\pi^\mu = \partial^\mu S_{\mathrm{HJ}}[A] \; , \qquad (7)$$

conveniently written in terms of the Hamilton-Jacobi action of the particle,

$$S_{\mathrm{HJ}} = p_0 \cdot x + \frac{e}{k \cdot p_0} \int^\phi d\phi' \, (p_0 \cdot A - eA^2/2) \; . \qquad (8)$$

On the other hand, we can see immediately that integrability is lost, *e.g.*, (i) for a standing wave, where $A^\mu = A^\mu(x^+, x^-)$, so that $\pi^- \neq const$, (the charge motion is known to be chaotic [2]), and (ii) for axially symmetric beams, where the loss of translation invariance in transverse directions ($\pi^i \neq const$) is compensated only incompletely by angular momentum conservation, $L_z = const$

PW: Orbits

Integrating the simple momentum dynamics (6) one finds that transverse and longitudinal dynamics separate, being linear and quadratic in A, respectively. Hence, along z, there is a zero mode, $\langle A^2 \rangle \sim a_0^2$, resulting in a longitudinal *drift* governed by the average or quasi momentum

$$q = p_0 + g\frac{m}{\omega_0} a_0^2 \, k \; , \; g = \begin{cases} 1 & \text{(L-polarisation)} \\ 2 & \text{(C-polarisation)} \end{cases} \qquad (9)$$

superimposed on the oscillatory PW behaviour. (ω_0 is the laser frequency in the electron rest frame.) The transverse motion basically follows the oscillations in A with a maximum excursion given by a_0 times the laser wavelength. In the average rest frame (no drift), one finds the well known figure-eight and circular orbits for linear and circular polarisation ($g = 1$ or 2), respectively [3]. For short, intense pulses, the quasi-momentum (9) becomes a rather elusive concept [4].

Example: PW + const

Adding a *constant* longitudinal electric field, ϵE, to a plane wave still yields a univariate potential, $A_\mu = A_\mu(x^-)$, but now $A^- = \epsilon E x^- \neq 0$. Here, E is the laser field amplitude and typically $\epsilon \ll 1$. This configuration can be used as a simplistic model for a laser propagating through a plasma [5] creating a space charge field ϵE.

Conservation of π^- implies $p^- = p_0^- \exp(\Omega\tau)$, with $\Omega := \epsilon a_0 \omega$, hence hyperbolic longitudinal dynamics. The transverse dynamics follows from conservation of the π^i. Its integration is more complicated, but can still be done analytically in terms of exponential sine and cosine functions, Si and Ci.

As expected intuitively, one essentially has plane wave orbits at short times and hyperbolic behaviour at large times, see Fig. 2 below. The onset of the latter moves towards earlier times with increasing ratio ϵ of longitudinal to laser electric field magnitudes.

PW + const: Orbits

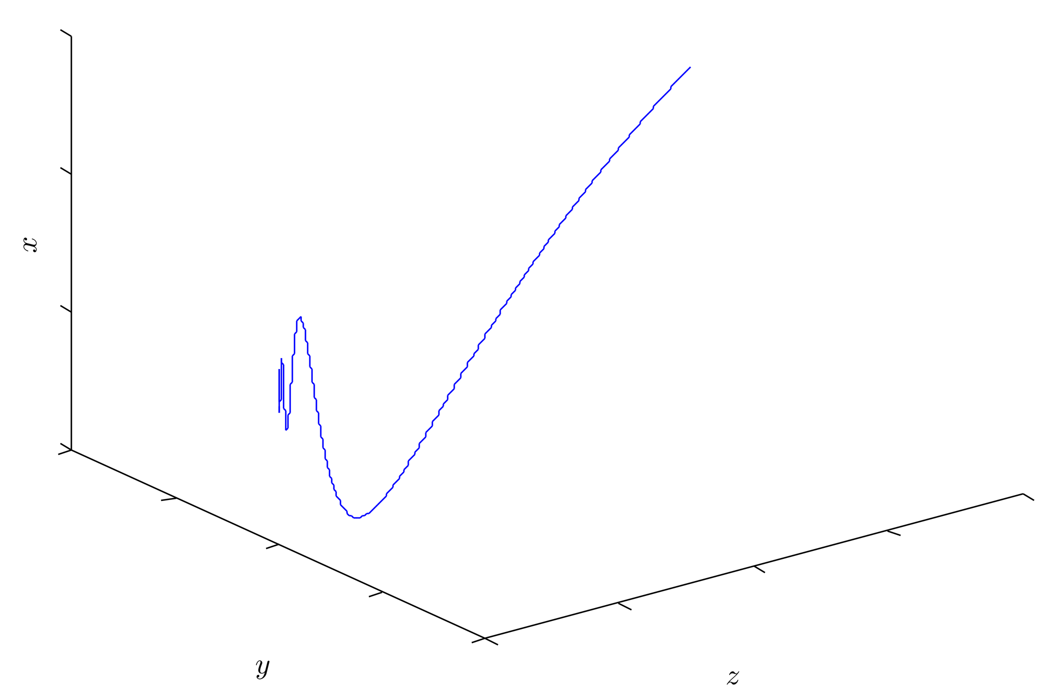

Fig. 2: Charge orbit in combined plane wave and longitudinal electric fields.

References

[1] A.H. Taub, *Orbits of Charged Particles in Constant Fields*. Phys. Rev. **73**, 786 (1948).

[2] D. Bauer, P. Mulser and W.-H. Steeb, *Relativistic Ponderomotive Force, Uphill Acceleration, and Transition to Chaos*, Phys. Rev. Lett. **75**, 4622 (1995).

[3] E.S. Sarachik and G.T. Schappert, *Classical Theory of the Scattering of Intense Laser Radiation by Free Electrons*, Phys. Rev. D **1**, 2738, (1970).

[4] C. Harvey, T. Heinzl, A. Ilderton and M. Marklund, *Intensity-Dependent Electron Mass Shift in a Laser Field: Existence, Universality, and Detection*, Phys. Rev. Lett. **109**, 100402 (2012).

[5] A.P.L. Robinson, A.V. Arefiev and D. Neely, *Generating "Superponderomotive" Electrons due to a Non-Wake-Field Interaction between a Laser Pulse and a Longitudinal Electric Field*, Phys. Rev. Lett. **111**, 065002 (2013).

LaTeX TikZposter